PHYS5150 — PLASMA PHYSICS

LECTURE 17 - MHD EQUILIBRIA

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- Variation of single-fluid equilibria
- Static: $\frac{\partial}{\partial t} = 0$ and $\bar{\mathbf{u}} = 0$
- Now MHD equations are:

$$\mathbf{j} \times \mathbf{B} = \nabla p$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$$

$$\nabla \cdot \mathbf{B} = 0$$

- Because of

$$\mathbf{j} \cdot (\mathbf{j} \times \mathbf{B}) = 0 = \mathbf{j} \cdot \nabla p$$

$$\mathbf{B} \cdot (\mathbf{j} \times \mathbf{B}) = 0 = \mathbf{B} \cdot \nabla p,$$

  $\mathbf{j}$ and $\mathbf{B}$ are lying on surfaces of constant pressure.

1 FORCE-FREE EQUILIBRIA

- cylindrical geometry
- current in θ direction
- $\mathbf{B}$ in z-direction

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Combine with MHD equations:

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} \quad \Big| \mathbf{B}\times$$

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \mu_0 \mathbf{B} \times \mathbf{j}$$

$$\nabla \left(\frac{B^2}{2} \right) - (\mathbf{B} \cdot \nabla)\mathbf{B} = -\mu_0 \underbrace{\mathbf{j} \times \mathbf{B}}_{\nabla p}$$

$$\nabla \left(\frac{B^2}{2} \right) + \mu_0 \nabla p = (\mathbf{B} \cdot \nabla)\mathbf{B}$$

$$\nabla \left\{ p + \frac{B^2}{2\mu_0} \right\} = \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla)\mathbf{B}$$

(Recall that we have found the same expression for the single-fluid case!). Now, for $\mathbf{B} = B_z(r)\hat{\mathbf{z}}$ this becomes

$$\frac{\mathrm{d}}{\mathrm{d}r} \left\{ p + \frac{B_z^2}{2\mu_0} \right\} = 0$$

and thus

$$p + \frac{B_z^2}{2\mu_0} = \text{constant}$$

if $p(\text{edge}) = 0$

$$p(r) + \frac{B_z(r)^2}{2\mu_0} = \underbrace{\frac{B_0^2}{2\mu_0}}_{\text{applied field}}$$

Note that $p_{total} = \text{constant}$.

1.1 *Plasma beta*

$$\nabla \left\{ \underbrace{p}_{①} + \underbrace{\frac{B_z^2}{2\mu_0}}_{②} \right\} = 0$$

Relative importance of particle and magnetic pressures:

$$\boxed{\beta = \frac{①}{②} = \frac{2\mu_0 p}{B^2}}$$

1.2 *Force-free plasmas*

- Applicable for low β, i.e. $\mathbf{j} \times \mathbf{B}$ dominates ∇p

- MHD equation is then $\mathbf{j} \times \mathbf{B} = 0$, i.e. $\mathbf{j} || \mathbf{B}$
- Means that $\mathbf{j}$ is field aligned
- Since $\mathbf{j} || \mathbf{B}$, Ampere's law can be written as $\mu_0 \mathbf{j} = \nabla \times \mathbf{B} = \alpha \mathbf{B}$, where the *lapse field* $\alpha(r)$ is a scalar function.

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \qquad \Big| \mathbf{B} \times$$

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = \mathbf{B} \times \alpha \mathbf{B}$$

$$\mathbf{B} \times (\nabla \times \mathbf{B}) = 0$$

- But also

$$\nabla \times \mathbf{B} = \alpha \mathbf{B} \qquad \Big| \nabla \cdot$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = \nabla \cdot (\alpha \mathbf{B})$$

  and recalling the vector identity $\nabla \cdot (\nabla \times \mathbf{f}) = 0$

$$0 = \nabla \cdot (\alpha \mathbf{B}) = \alpha \underbrace{(\nabla \cdot \mathbf{B})}_{0} + \mathbf{B} \cdot \nabla \alpha$$

$$\mathbf{B} \cdot \nabla \alpha = 0$$

  This implies that $\mathbf{B}$ lies on constant-α surfaces!

- Constant-α surfaces? Possible topologies could be spheres, doughnuts, toroids, etc.
- Such a surface cannot be simple and closed:
  - Assume that this would be the case and consider closed curve along a field line:
  $$\int_C \mathbf{B} \cdot d\mathbf{l} \neq 0 = \int_S (\nabla \times \mathbf{B})\, d\mathbf{A} = \int_S \alpha \mathbf{B}\, d\mathbf{A} \neq 0$$
  - Now distort surface S such that it lies on a constant-α surface
  - Now pull α out of the integral because it is constant on such a surface
  $$\int_C \mathbf{B} \cdot d\mathbf{l} = \alpha \int_S \mathbf{B}\, d\mathbf{A} \neq 0$$
  - Contradiction to $\mathbf{B} \cdot \nabla \alpha = 0$, so that our assumption of a constant-α surface is "simply closed" is wrong.
- Hopf's theorem shows that those surfaces must be toroidal (in simplest form)